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Integrable and chaotic motions of four vortices
I. The case of identical vortices

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It is shown that the three-vortex problem in two-dimensional hydrodynamics is integrable, whereas the motion of four identical vortices is not. A sequence of canonical transformations is obtained that reduces the N-degree-of-freedom Hamiltonian, which describes the interaction of N identical vortices, to one with $N-2$ degrees of freedom. For $N = 3$ a reduction to a single degree of freedom is obtained and this problem can be solved in terms of elliptic functions. For $N = 4$ the reduction procedure leads to an effective Hamiltonian with two degrees of freedom of the form found in problems with coupled nonlinear oscillators. Resonant interaction terms in this Hamiltonian suggest non-integrable behaviour and this is verified by numerical experiments. Explicit construction of a solution that corresponds to a heteroclinic orbit in phase space is possible. The relevance of the results obtained to fundamental problems in hydrodynamics, such as the question of integrability of Euler's equation in two dimensions, is discussed. The paper also contains a general exposition of the Hamiltonian and Poisson-bracket formalism for point vortices.

1. INTRODUCTION

We present analytical and numerical results on the motion of four point vortices in a plane. Our study includes three different cases: (i) the motion of four identical vortices, (ii) the advection of a passive marker particle (a 'vortex' of strength zero) by three (identical) vortices and (iii) the collision of two neutral vortex pairs. The present paper is concerned with problem (i) and also contains a review of the general theory of point vortex dynamics. In a sequel, currently in preparation, problems (ii) and (iii) will be considered. We show here that the motion of four identical vortices displays chaotic or stochastic solutions. We identify the source and discuss the implications of stochastic behaviour.

The work reported in this study, short accounts of which have appeared elsewhere (Aref & Pomphrey 1980*a*, *b*), relates to several currently active areas of theoretical mechanics including the theory of turbulence. There is an obvious connection with the theory of dynamical systems (see the entire volume edited by Jorna (1978)) as applied, for example, to the three-body problem in celestial mechanics, and the methods and terminology of that theory will be used throughout the paper. There is a more tenuous connection with the theory of two-dimensional turbulence (see the review by Kraichnan & Montgomery 1980). There are analogies (in the collision dynamics of vortex pairs) to the theory of solitons (see Scott *et al.* 1973).

In the widest possible definition two-dimensional turbulence comprises all stochastic solutions of the equations of motion of two-dimensional hydrodynamics. As we shall see, four point vortices suffice to produce such stochastic solutions. On the other hand, any kind of fully developed turbulence of necessity involves more than four degrees of freedom. Thus, it is more accurate to describe the stochasticity in four-vortex motion as a 'transition' to statistical mechanics as the number of degrees of freedom is increased. Although stochasticity can be achieved in both dissipative (Lorenz 1963) and conservative (Hénon & Heiles 1964) systems with a few degrees of freedom, any distinctions between equilibrium and non-equilibrium behaviour involve the limit of many degrees of freedom. In particular any notion of a turbulent cascade in the statistical dynamics of four vortices would appear misplaced.

This is not to say that the results obtained are of no interest to turbulence theory. The fact that four-vortex motion can display chaotic behaviour in some region of phase space suggests a new view of the predictability problem (as mentioned briefly by Novikov & Sedov 1978). Traditionally the arguments for unpredictability of two-dimensional fluid flow involve the notion that inaccurate knowledge of small-scale motions cascades to larger scales in a finite time (see Lorenz 1969; Leith & Kraichnan 1972). The results here suggest that matters may be much worse: there are already régimes in the phase space of the large scales where the flow is intrinsically unpredictable. Inaccuracies do not have to propagate from small scales.

Previous work on the four-vortex problem by Novikov & Sedov (1978, 1979*b*) and Inogamov & Manakov (1979) has emphasized connections with the literature on integrable systems and solitons by phrasing the question: 'Is two-dimensional, inviscid, incompressible hydrodynamics an integrable dynamical system?' In the papers cited the argument advanced is (i) that the four-vortex problem can be considered to be embedded in the full continuum fluid equations, (ii) that the integrability or non-integrability of four-vortex motion can be decided by the methods of analytical dynamics and finally (iii) that stochasticity in the four-vortex problem implies *a fortiori* stochasticity of two-dimensional (inviscid, incompressible) flow in general. Although this line of reasoning seems basically sound, it should be stressed that the point-vortex problem is intrinsically more singular than the continuum problem. In particular, there are initial configurations for three point vortices that lead to a singularity after a finite time (the vortices collide at a point). These motions (for details see Aref (1979); for generalizations to more than three vortices see Novikov & Sedov (1979*a*)) have no counterpart in the dynamics of a smooth distribution of vorticity owing to a classical theorem by Wolibner (1933) (see the review by Rose & Sulem (1978) for discussion). This simple observation suggests that steps (i) and (iii) of the above argument need clarification. The alternative would be that the N-vortex problem is non-integrable for every N, $4 \leqslant N < \infty$, but that the Euler equation, which presumably arises in the limit $N \rightarrow \infty$, remains integrable. (We are indebted to Professor A. J. Chorin for comments on this point.) As an example of an analogous situation, recall that if the exponential interactions between particles in the *integrable* three-body Toda lattice are expanded in a power series, one obtains by truncating this expansion at successive orders a sequence of Hamiltonians, the first non-trivial one being the non-integrable Hénon & Heiles system (see the article by M. V. Berry in Jorna (1978)). In § 7 we return to a discussion of integrability of the Euler equation in two dimensions.

The remaining sections of our paper are organized as follows: In § 2 we review the dynamics of point vortices and develop the Hamiltonian formalism. We prove anew that the three-vortex problem is integrable for arbitrary vortex strengths. In § 3 we provide a sequence of canonical transformations that reduces the problem of N identical vortices to one with $N-2$ degrees of freedom. In § 4 we use these transformations to reduce the three-vortex problem to one with a single degree of freedom. A solution of this problem in terms of Jacobi elliptic functions is obtained. In § 5 the problem of four identical vortices is reduced to one with two degrees of freedom. The effective Hamiltonian of the reduced problem can be interpreted as the interaction of two nonlinear oscillators with resonant couplings. Such systems can produce stochastic solutions by a mechanism known as 'resonance overlap' (Chirikov (1979); see also the article by J. Ford in Jorna (1978)). A family of exact solutions that qualitatively correspond closely to the motions of three vortices are discussed. Among these are some that correspond to heteroclinic orbits of the reduced Hamiltonian. In § 6 we present results of numerical simulations of four-vortex motion. We calculate phase trajectories and Poincaré sections for both the (apparently) integrable and chaotic régimes. Some other quantities and diagnostics are also discussed. Finally § 7 contains a discussion of the results obtained.

2. The dynamics of point vortices

In this section we review the dynamics of a system of point vortices and establish our notation (compare the treatments by Lamb (1945); Sommerfeld (1964); Friedrichs (1966) and Batchelor (1967)). We identify the flow plane with the complex z-plane and describe each vortex $\alpha = 1, \dots, N$ by a position $z_\alpha = x_\alpha + \mathrm{i} y_\alpha$ and a strength κ_α. The strengths (or circulations) are constant in time. The motion of the point vortices is governed by the equations

$$\dot{z}_\alpha^* = \frac{1}{2\pi \mathrm{i}} \sum_{\beta=1}^{N}{}' \frac{\kappa_\beta}{z_\alpha - z_\beta}; \quad \alpha = 1, \dots, N, \tag{2.1}$$

where the asterisk denotes complex conjugation, and the prime signifies that the singular term $\beta = \alpha$ is omitted from the sum. A point vortex does not contribute to its own velocity. Equation (2.1) follow immediately from Helmholtz's vorticity theorems. They define a many-body problem of remarkable formal simplicity.

Kirchhoff (1883) noticed that equation (2.1) may be written as

$$\kappa_\alpha \dot{x}_\alpha = \frac{\partial H}{\partial y_\alpha}, \quad \kappa_\alpha \dot{y}_\alpha = -\frac{\partial H}{\partial x_\alpha}, \tag{2.2}$$

where

$$H = -\frac{1}{4\pi} \sum_{\alpha,\beta}{}' \kappa_\alpha \kappa_\beta \ln |z_\alpha - z_\beta|. \tag{2.3}$$

These may be brought entirely into Hamiltonian form by defining

$$q_\alpha = \kappa_\alpha^{\frac{1}{2}} x_\alpha, \qquad p_\alpha = \kappa_\alpha^{\frac{1}{2}} y_\alpha \quad \text{for} \quad \kappa_\alpha > 0, \tag{2.4a}$$

$$q_\alpha = |\kappa_\alpha|^{\frac{1}{2}} y_\alpha, \quad p_\alpha = |\kappa_\alpha|^{\frac{1}{2}} x_\alpha \quad \text{for} \quad \kappa_\alpha < 0. \tag{2.4b}$$

Written in terms of the q_α and p_α, equations (2.1) become Hamilton's canonical equations with q_α, p_α canonically conjugate variables for $\alpha = 1, \ldots, N$ and H the Hamiltonian. The system has as many degrees of freedom as there are vortices. We note that the definitions (2.4) are not unique. Any linear relation

$$\begin{bmatrix} q_\alpha \\ p_\alpha \end{bmatrix} = \begin{bmatrix} a_{11}^{(\alpha)} & a_{12}^{(\alpha)} \\ a_{21}^{(\alpha)} & a_{22}^{(\alpha)} \end{bmatrix} \begin{bmatrix} x_\alpha \\ y_\alpha \end{bmatrix}, \tag{2.5}$$

with the property

$$\det \begin{bmatrix} a_{11}^{(\alpha)} & a_{12}^{(\alpha)} \\ a_{21}^{(\alpha)} & a_{22}^{(\alpha)} \end{bmatrix} = \kappa_\alpha \tag{2.6}$$

will do. In our definition the canonically conjugate variables have the same physical units.

The Hamiltonian (2.3) is invariant under the continuous transformation group of translations and rotations of the coordinates. These symmetries yield by well known methods the integrals

$$\sum_\alpha \kappa_\alpha z_\alpha \equiv Q + \mathrm{i}P \tag{2.7}$$

(where Q, P are real) and

$$\sum_\alpha \kappa_\alpha |z_\alpha|^2 \equiv L^2. \tag{2.8}$$

The equations of motion (2.1) are also invariant under certain discrete transformations. For example if (in an easily understood notation) $\{\kappa_\alpha, z_\alpha\}$ is a solution of (2.1) then so is $\{\kappa_\alpha, -z_\alpha\}$ and $\{-\kappa_\alpha, z_\alpha^*\}$. These discrete symmetries do not yield generally conserved quantities like (2.7) and (2.8) but, as is easily seen, they do have the property that if they are satisfied by the initial configuration they are preserved for all future times. For example, if we initially have a configuration of four vortices with $\kappa_1 = \kappa_2 = -\kappa_3 = -\kappa_4$ and $z_4 = z_1^*, z_3 = z_2^*$ then these relations are obeyed for all future times. Such symmetries will lead to analytically solvable cases of four-vortex motion, a property already used by Love (1894). We shall make use of a discrete symmetry of this type in § 5b.

To proceed with the formal development we define the Poisson bracket $[f, g]$ of two quantities f and g depending on the vortex positions (and strengths) by (cf. Landau & Lifshitz 1969)

$$[f, g] \equiv \sum_{\alpha=1}^{N} \left(\frac{\partial f}{\partial q_\alpha} \frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha} \frac{\partial g}{\partial q_\alpha} \right). \tag{2.9}$$

It is then easy to show, with the aid of (2.4) (or in fact any of the possibilities (2.5)), that

$$[f, g] \equiv \sum_{\alpha=1}^{N} \frac{1}{\kappa_\alpha} \left(\frac{\partial f}{\partial x_\alpha} \frac{\partial g}{\partial y_\alpha} - \frac{\partial f}{\partial y_\alpha} \frac{\partial g}{\partial x_\alpha} \right). \tag{2.10}$$

The fundamental Poisson brackets are

$$[z_\alpha, z_\beta] = 0, \quad [z_\alpha, z_\beta^*] = -2\mathrm{i}\delta_{\alpha\beta}/\kappa_\alpha. \tag{2.11}$$

We next consider the Poisson-bracket algebra for the conserved quantities P, Q and L^2 characterized by

$$[H, P] = [H, Q] = [H, L^2] = 0. \tag{2.12}$$

Using (2.11) we obtain

$$[Q, P] = \sum_\alpha \kappa_\alpha, \tag{2.13a}$$

$$[Q, L^2] = 2P, \tag{2.13b}$$

$$[P, L^2] = -2Q, \tag{2.13c}$$

whence

$$[P^2 + Q^2, L^2] = 0. \tag{2.14}$$

It follows that H, L^2 and $P^2 + Q^2$ are analytic integrals in involution regardless of the values of the vortex strengths (cf. Novikov & Sedov 1978). Since a Hamiltonian system with N degrees of freedom is integrable if it has N integrals in involution (see Whittaker 1937), we are guaranteed that the motion of three vortices is integrable for any values of the vortex strengths. This result was known to Kirchhoff (1883) and to Poincaré (1893) but a full elucidation of the general properties of three-vortex motion is more recent (Novikov 1975; Aref 1979). We shall return to the motion of three identical vortices in § 4.

In the solution of the problem of three identical vortices the ratio of arithmetic to geometric mean of the vortex separations is a decisive parameter (Novikov 1975). For arbitrary N we define

$$\Theta \equiv \left[\frac{1}{N(N-1)} \sum_{\alpha,\beta=1}^{N} |z_\alpha - z_\beta|^2 \right]^{\frac{1}{2}N(N-1)} \bigg/ \prod_{\alpha,\beta=1}^{N}{}' \, |z_\alpha - z_\beta|. \tag{2.15}$$

It is easy to show that if the origin of coordinates is chosen as the position of the centre of vorticity, i.e. if $P = Q = 0$ in (2.7), and units are chosen such that all the (identical) κ_α are unity, we have

$$\sum_{\alpha,\beta=1}^{N} |z_\alpha - z_\beta|^2 = 2NL^2. \tag{2.16}$$

Furthermore, from (2.3)

$$\prod_{\alpha,\beta=1}^{N}{}' \, |z_\alpha - z_\beta| = \mathrm{e}^{-4\pi H}. \tag{2.17}$$

Thus

$$\Theta = \left(\frac{2}{N-1}\right)^{\frac{1}{2}N(N-1)} L^{N(N-1)}\, \mathrm{e}^{4\pi H}. \tag{2.18}$$

In particular for $N = 3$

$$\Theta = L^6\, \mathrm{e}^{4\pi H} \tag{2.19a}$$

and for $N = 4$

$$\Theta = (\tfrac{2}{3})^6 L^{12}\, \mathrm{e}^{4\pi H}. \tag{2.19b}$$

By the Cauchy–Schwarz inequality $\Theta \geqslant 1$. For $N = 3$ this minimum value is attained for an equilateral-triangle configuration. For $N = 4$ the minimum value of Θ is $\Theta_0^{(4)} = 2^{10}/3^6 = 1.40466$ attained for a square configuration. The problem of determining the minimum value $\Theta_0^{(N)}$ of Θ for arbitrary N is related to the wider problem of finding uniformly rotating configurations with N identical vortices (cf. Campbell & Ziff 1979). We shall address this problem in a separate study. For use in later sections we define

$$\Lambda \equiv (\Theta_0^{(N)}/\Theta)^{\frac{1}{2}}, \tag{2.20}$$

which lies in the range $0 < \Lambda \leqslant 1$.

3. Reduction of degrees of freedom by canonical transformations

We have alluded to the important role played by the isolating integrals L^2 and $P^2 + Q^2$: their existence guarantees integrability of a system with three vortices. In this section we use these integrals to construct canonical transformations that greatly simplify the discussion of the dynamics of a small number of identical vortices. In general these transformations reduce by two the number of effective degrees of freedom whatever the number of vortices. The use of analytic integrals for reducing the number of degrees of freedom of a Hamiltonian system is discussed by Synge (1960). If $C(q,p)$ is a known integral for a Hamiltonian $H(q,p)$, a Hamilton–Jacobi differential equation is constructed from $C(q,p)$ by replacing the momentum, p, by $\partial S(q,P)/\partial q$. Then $S(q,P)$ is a solution of the Hamilton–Jacobi equation that generates a canonical transformation to new coordinates and momenta (Q,P) such that one of the new momenta has the constant value C. Its conjugate variable is therefore cyclic (absent) in the transformed Hamiltonian, $H(Q,P)$, and a reduction of degrees of freedom by one is accomplished.

To apply this method to the vortex dynamics of § 2 is in principle straightforward since both the integrals L^2 and $P^2 + Q^2$ are just quadratic in coordinates and momenta. For identical vortices of unit strength (which we shall consider from now on)

$$L^2 = \sum_{\alpha=1}^{N} (q_\alpha^2 + p_\alpha^2), \tag{3.1a}$$

$$P^2 + Q^2 = \sum_{\alpha,\beta=1}^{N} (q_\alpha q_\beta + p_\alpha p_\beta). \tag{3.1b}$$

The actual construction and solution of the Hamilton–Jacobi equation associated with equation (3.1) can, however, be pre-empted by observing that a simple rotation of coordinates,

$$Q_n + \mathrm{i}P_n = N^{-\frac{1}{2}} \sum_{\alpha=1}^{N} \mathrm{e}^{\mathrm{i}(2\pi n/N)(\alpha-1)} z_\alpha; \quad n = 0, 1, \ldots, N-1, \tag{3.2}$$

diagonalizes both quadratic forms. We note that (3.2) is a discrete Fourier transform of the positions thought of as an array of complex data. (This transformation was suggested to us by Dr R. G. Littlejohn.) It is easily verified and anticipated in the notation that the variables Q_n, P_n are canonically conjugate:

$$[Q_n + \mathrm{i}P_n, Q_l + \mathrm{i}P_l] = 0, \tag{3.3a}$$

$$[Q_n + \mathrm{i}P_n, Q_l - \mathrm{i}P_l] = 2\mathrm{i}\delta_{ln}. \tag{3.3b}$$

Furthermore, we see that

$$Q_0 + \mathrm{i}P_0 = N^{-\frac{1}{2}}(Q + \mathrm{i}P) \tag{3.4}$$

(whence $Q^2 + P^2 = N(Q_0^2 + P_0^2)$). Thus Q_0 and P_0 are both constants of the motion and will not appear in the transformed Hamiltonian. We may choose $Q_0 = P_0 = 0$ with no loss of generality. By Parseval's theorem, which expresses the conservation of norm under the discrete Fourier transform (3.2), we then see that

$$\sum_{n=1}^{N} Q_n^2 + P_n^2 = L^2. \tag{3.5}$$

The transformation inverse to (3.2) is given by

$$z_\alpha = N^{-\frac{1}{2}} \sum_{n=0}^{N-1} \mathrm{e}^{-\mathrm{i}(2\pi n/N)(\alpha-1)} (Q_n + \mathrm{i}P_n). \tag{3.6}$$

Next introduce polar coordinates or 'action-angle' variables J_n, θ_n through

$$(2J_n)^{\frac{1}{2}} \mathrm{e}^{\mathrm{i}\theta_n} = Q_n + \mathrm{i}P_n; \quad n = 1, \ldots, N-1. \tag{3.7}$$

The remaining integral is *linear* in the new action variables:

$$\sum_{n=1}^{N-1} J_n = \tfrac{1}{2}L^2. \tag{3.8}$$

We also notice that since the Hamiltonian depends only on the separations $|z_\alpha - z_\beta|$ it may be written entirely in terms of $J_1, \ldots, J_{N-1}$ and the angle *differences*

$\theta_1-\theta_{N-1}, \ldots, \theta_{N-2}-\theta_{N-1}$. This suggests a final transformation to a new set of canonical variables $I_1, \ldots, I_{N-1}, \phi_1, \ldots, \phi_{N-1}$ so that

$$I_{N-1} = \sum_{n=1}^{N-1} J_n. \tag{3.9}$$

We use a generating function of type F_3 (Goldstein 1950):

$$F_3(J_1, \ldots, J_{N-1}, \phi_1, \ldots, \phi_{N-1}) \equiv \sum_{k,l=1}^{N-1} C_{kl} J_k \phi_l, \tag{3.10}$$

where the C_{kl} make up a regular matrix of constant coefficients. Apart from the further restriction that all $C_{k,N-1}$ be equal, there is some flexibility in the choice of these coefficients. We shall see in §§ 4 and 5 that specific choices lead to variables of simple geometrical significance.

The equations of transformation associated with (3.10) are

$$I_k = \sum_{l=1}^{N-1} C_{lk} J_l, \tag{3.11a}$$

$$\theta_k = \sum_{l=1}^{N-1} C_{kl} \phi_l. \tag{3.11b}$$

The angle ϕ_{N-1} is cyclic since I_{N-1} is conserved.

In summary, we have achieved for the problem of N identical vortices a reduction by successive canonical transformations to one with $N-2$ degrees of freedom. This is particularly useful for $N = 3$ and $N = 4$.

4. The case $N = 3$: motion of three identical vortices

We now present details of a solution for the case $N = 3$ that uses the canonical transformations of § 3. This problem was solved recently by Novikov (1975) using a graphical method. Our analysis confirms essentially all of these earlier results but it also leads immediately to explicit expressions (involving Jacobi elliptic functions and complete elliptic integrals) for many of the quantities that Novikov (1975) could only give as integral formulae. Although our equations and results could be obtained by reworking the equations in Novikov's paper, the algebra involved in such a derivation appears very tedious. In any event the derivation presented here provides a useful example of the general formalism in § 3.

We first use the transformation equations (3.2) and (3.7) and then introduce

$$F_3(J_1, J_2, \phi_1, \phi_2) = \tfrac{1}{2}J_2(\phi_2+\phi_1) + \tfrac{1}{2}J_1(\phi_2-\phi_1), \tag{4.1}$$

so that

$$I_1 = \tfrac{1}{2}(J_2-J_1), \quad I_2 = \tfrac{1}{2}(J_2+J_1), \tag{4.2a}$$

$$\phi_1 = \theta_2-\theta_1, \quad \phi_2 = \theta_2+\theta_1. \tag{4.2b}$$

The problem is now reduced to a one-degree-of-freedom Hamiltonian $H(I_1, \phi_1)$. The merit of the transformation (4.2) is that I_1 has a simple geometrical interpretation. Indeed, it is easily shown that

$$I_1 = \tfrac{1}{\sqrt{3}}\,\sigma_{123} A_{123}, \tag{4.3}$$

where σ_{123} is the orientation (+ 1 for 123 arranged clockwise) and A_{123} is the area of the triangle spanned by the three vortices.

Using the transformation formulae, equation (3.6), with a straightforward calculation we derive that

$$H(I_1, \phi_1) = -(4\pi)^{-1}\ln\{16[I_2(I_2^2+3I_1^2)-(I_2^2-I_1^2)^{\frac{3}{2}}\cos(3\phi_1)]\} \tag{4.4}$$

where $I_2 = \frac{1}{4}L^2$ is a constant. We shall omit the details of the derivation of (4.4) since the analogous but more complicated calculation for $N = 4$ is displayed in § 5.

The equation of motion for I_1 then is

$$\dot{I}_1 = -\frac{\partial H}{\partial \phi_1} = \frac{12\mathrm{e}^{4\pi H}}{\pi}(I_2^2-I_1^2)^{\frac{3}{2}}\sin(3\phi_1). \tag{4.5}$$

Squaring both sides of (4.5) and using (2.19a) and (2.20) we obtain

$$(\mathrm{d}I/\mathrm{d}\tau)^2 = -I[I^3+6I^2+3(3-8\Lambda^2)I+8\Lambda^2(2\Lambda^2-1)], \tag{4.6}$$

where

$$I \equiv (I_1/I_2)^2, \tag{4.7}$$

and

$$\tau = \frac{3}{2\pi}\frac{\kappa}{\Lambda^2 L^2}t \tag{4.8}$$

is a scaled time variable.

The roots of the cubic on the right-hand side of (4.6) are always real and are given by

$$I^{(n)} = 2\{(1+8\Lambda^2)^{\frac{1}{2}}\cos[\tfrac{1}{3}(2n\pi+\delta)]-1\}, \quad n = 0, 1, 2, \tag{4.9}$$

where

$$\cos\delta = (-8\Lambda^4-20\Lambda^2+1)/(1+8\Lambda^2)^{\frac{3}{2}}. \tag{4.10}$$

As Λ varies from 0 to 1, δ varies from 0 to π. We always have $I^{(1)} < I^{(2)} < I^{(0)}$. The root $I^{(2)}$ increases from -3 (when $\Lambda = 0$) to 1 (when $\Lambda = 1$) crossing 0 when $\Lambda = \frac{1}{\sqrt{2}}$ If we let

$$\mathscr{I}_0 = I^{(0)}, \tag{4.11a}$$

$$\mathscr{I}_1 = \max\{0, I^{(2)}\}, \tag{4.11b}$$

$$\mathscr{I}_2 = \min\{0, I^{(2)}\}, \tag{4.11c}$$

$$\mathscr{I}_3 = I^{(1)} \tag{4.11d}$$

then $\mathscr{I}_3 < \mathscr{I}_2 < \mathscr{I}_1 < \mathscr{I}_0$ and $\mathscr{I}_1 \leqslant I \leqslant \mathscr{I}_0$ during the motion. The solution to equation (4.6) may be written (cf. Byrd & Friedman 1971)

$$I = \frac{\mathscr{I}_0 - \mathscr{I}_3\alpha^2\,\mathrm{sn}^2(\omega\tau)}{1-\alpha^2\,\mathrm{sn}^2(\omega\tau)}, \tag{4.12}$$

where

$$\alpha^2 \equiv (\mathscr{I}_1 - \mathscr{I}_0)/(\mathscr{I}_1 - \mathscr{I}_3), \tag{4.13}$$

$$\omega \equiv \tfrac{1}{2}[(\mathscr{I}_0 - \mathscr{I}_2)(\mathscr{I}_1 - \mathscr{I}_3)]^{\frac{1}{2}}, \tag{4.14}$$

and the modulus of the Jacobi elliptic function is

$$m^2 = (\mathscr{I}_0 - \mathscr{I}_1)(\mathscr{I}_2 - \mathscr{I}_3)/(\mathscr{I}_0 - \mathscr{I}_2)(\mathscr{I}_1 - \mathscr{I}_3). \tag{4.15}$$

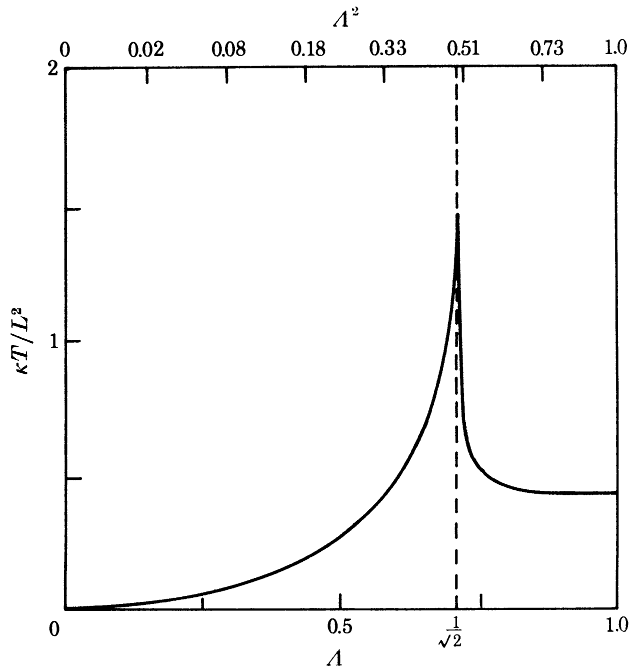

FIGURE 1. The dimensionless period $\kappa T/L^2$, equation (4.19), against Λ (bottom) or Λ^2 (top). The relative motion of three identical vortices has period T for $\Lambda < \frac{1}{\sqrt{2}}$, period $3T$ for $\frac{1}{\sqrt{2}} < \Lambda < 1$.

For the period of the pulsations of the area variable I we have

$$T = \frac{4\pi}{3}\frac{\Lambda^2 L^2}{\kappa}\frac{K(m)}{\omega} \times \begin{cases} 2, & \text{for} \quad 0 < \Lambda < \frac{1}{\sqrt{2}}, \\ 1, & \text{for} \quad \frac{1}{\sqrt{2}} < \Lambda < 1, \end{cases} \tag{4.16}$$

where K is the complete elliptic integral of the first kind. For $\Lambda = \frac{1}{\sqrt{2}}$ the vortex triangle collapses to a collinear configuration. This motion is aperiodic. In the present variables it is given by the simple formula

$$I = \left[1 + \frac{2}{\sqrt{3}}\cosh(\tau\sqrt{3})\right]^{-1}. \tag{4.17}$$

A general qualitative discussion of the three régimes $0 < \Lambda < \frac{1}{\sqrt{2}}$, $\Lambda = \frac{1}{\sqrt{2}}$ and $\frac{1}{\sqrt{2}} < \Lambda \leqslant 1$ appears in the earlier papers (Novikov 1975; Aref 1979) and will not be repeated here. We include in figure 1 a plot of the dimensionless period $\kappa T/L^2$ against Λ. A similar plot for $N = 4$ appears as figure 8. The period has a (logarithmic) singularity at the 'critical' value $\Lambda = \Lambda_c^{(3)} \equiv \frac{1}{\sqrt{2}}$. The period for the relative motion equals T, equation (4.19), for $0 < \Lambda < \frac{1}{\sqrt{2}}$ but equals $3T$ for $\frac{1}{\sqrt{2}} < \Lambda < 1$. (The period for relative motion must clearly always be a multiple of the period for pulsations in the area variable I). The resulting expressions have the correct limits in the explicitly calculable cases $\Lambda \to 0$ (a single pair far removed from the third vortex) and $\Lambda \to 1$ (small pulsations of an equilateral triangle; it is a classical result that this period equals the period of rotation of the triangle).

5. The case $N = 4$: motion of four identical vortices

By using the technique of § 3 the four-vortex Hamiltonian may be reduced to one with two degrees of freedom. This section provides details of the reduction process (§ 5*a*) and also a discussion of a special class of solutions including some that correspond to heteroclinic points in phase space (§ 5*b*). The form of the Hamiltonian, equations (5.8)–(5.12), has been available to us for some time and was discussed in our earlier paper (Aref & Pomphrey 1980*a*). Meanwhile Khanin (1980) has derived independently a reduction of the four-vortex problem to one with two degrees of freedom. His derivation does not immediately generalize to $N > 4$ but it does accommodate the case where all vortex strengths are of one sign without being of equal magnitude. The case of $N = 2n$ vortices, n with the strength $+\kappa$ and n with strength $-\kappa$, has also been reduced to one with $N-2$ degrees of freedom. Details for this case will appear in a forthcoming paper on the collision dynamics of vortex pairs.

(*a*) *Derivation of the Hamiltonian*

Let $Q_n + \mathrm{i}P_n$, $n = 0, 1, 2, 3$, (see § 3) be denoted by ζ_n and assume $\zeta_0 = 0$ without loss of generality. Then

$$z_1 = \tfrac{1}{2}(\zeta_1 + \zeta_2 + \zeta_3), \qquad z_2 = \tfrac{1}{2}(-\mathrm{i}\zeta_1 - \zeta_2 + \mathrm{i}\zeta_3), \tag{5.1 a, b}$$

$$z_3 = \tfrac{1}{2}(-\zeta_1 + \zeta_2 - \zeta_3), \quad z_4 = \tfrac{1}{2}(\mathrm{i}\zeta_1 - \zeta_2 - \mathrm{i}\zeta_3). \tag{5.1 c, d}$$

It is easy to derive from these equations that

$$(z_1 - z_3)(z_2 - z_4) = \mathrm{i}(\zeta_3^2 - \zeta_1^2), \tag{5.2 a}$$

$$(z_1 - z_2)(z_1 - z_4)(z_2 - z_3)(z_3 - z_4) = \tfrac{1}{4}[8\zeta_1\zeta_2^2\zeta_3 - 4\zeta_2^4 - (\zeta_1^2 + \zeta_3^2)^2]. \tag{5.2 b}$$

We consider the generating function

$$F_3(J_1, J_2, J_3, \phi_1, \phi_2, \phi_3) = \phi_1(J_1 - J_3) + \phi_2(J_1 + J_3) + \phi_3(J_1 + J_2 + J_3). \tag{5.3}$$

The corresponding equations of transformation are

$$\phi_1 = \tfrac{1}{2}\theta_1 - \tfrac{1}{2}\theta_3; \qquad \theta_1 = \phi_1 + \phi_2 + \phi_3; \tag{5.4 a}$$

$$\phi_2 = \tfrac{1}{2}\theta_1 - \theta_2 + \tfrac{1}{2}\theta_3; \quad \theta_2 = \phi_3; \tag{5.4 b}$$

$$\phi_3 = \theta_2; \qquad \theta_3 = -\phi_1 + \phi_2 + \phi_3; \tag{5.4 c}$$

and

$$I_1 = J_1 - J_3; \qquad J_1 = \tfrac{1}{2}I_1 + \tfrac{1}{2}I_2; \tag{5.5a}$$

$$I_2 = J_1 + J_3; \qquad J_2 = -I_2 + I_3; \tag{5.5b}$$

$$I_3 = J_1 + J_2 + J_3; \quad J_3 = -\tfrac{1}{2}I_1 + \tfrac{1}{2}I_2. \tag{5.5c}$$

Note that $I_1 \leqslant I_2 \leqslant I_3$. From these definitions it is easily seen that

$$I_2 = \tfrac{1}{4}(|z_1 - z_3|^2 + |z_2 - z_4|^2). \tag{5.6a}$$

Furthermore if subscript zero corresponds to the origin (centre of vorticity)

$$I_1 = \sigma_{012}A_{012} + \sigma_{023}A_{023} + \sigma_{034}A_{034} + \sigma_{041}A_{041}, \tag{5.6b}$$

where (see equation (4.3)) $A_{0\mu\lambda}$ is the area of the triangle with vertices at the origin, z_μ and z_λ, and $\sigma_{0\mu\lambda}$ is its orientation. If the quadrilateral spanned by the four vortices is convex and the vortices appear in clockwise order, I_1 is clearly the area of the quadrilateral and I_2 is one quarter of the sum of squares of the diagonals.

We substitute for ζ_n, $n = 1, 2, 3$, its expression in terms of J_n and θ_n, $\zeta_n = (2J_n)^{\frac{1}{2}} e^{i\theta_n}$, and then substitute for θ_1, θ_2 and θ_3 their expressions in terms of ϕ_1, ϕ_2, ϕ_3. After factoring out a term $2i\, e^{6i(\phi_2+\phi_3)}$ we obtain

$$\begin{aligned}\prod_{1\leqslant k<l\leqslant 4} |z_k - z_l| = 2|\{&J_1J_3(J_1-J_3)\cos 2\phi_1 + (J_1^3 - J_3^3)\cos 6\phi_1 + 4J_1J_2^2\cos(2\phi_1 - 4\phi_2)\\ &- 4J_2^2J_3\cos(2\phi_1 + 4\phi_2) - 8J_1^{\frac{3}{2}}J_2J_3^{\frac{1}{2}}\cos(2\phi_1 - 2\phi_2)\\ &+ 8J_1^{\frac{1}{2}}J_2J_3^{\frac{3}{2}}\cos(2\phi_1 + 2\phi_2)\\ &+ i[(J_1^3 + J_3^3)\sin 6\phi_1 + J_1J_3(J_1+J_3)\sin 2\phi_1\\ &+ 4J_1J_2^2\sin(2\phi_1 - 4\phi_2) + 4J_2^2J_3\sin(2\phi_1 + 4\phi_2)\\ &- 8J_1^{\frac{3}{2}}J_2J_3^{\frac{1}{2}}\sin(2\phi_1 - 2\phi_2) - 8J_1^{\frac{1}{2}}J_2J_3^{\frac{3}{2}}\sin(2\phi_1 + 2\phi_2)]\}|.\end{aligned}$$

The modulus on the right-hand side involves terms that are products of at most two trigonometric functions with coefficients depending on the J_n. Thus the whole expression may be written as a sum of terms of the form $f_{m,n}(J_1, J_2, J_3) \times \cos(m\phi_1 + n\phi_2)$ where m, n are integers. The $f_{m,n}$ are algebraic functions of J_1, J_2 and J_3. Using the transformation formulae (5.5) we may reexpress them in terms of I_1, I_2 and I_3. We shall not reproduce here the tedious algebraic manipulations effecting these transformations but simply write the final Hamiltonian:

$$H(I_1, I_2, \phi_1, \phi_2) = -(4\pi)^{-1}\ln h(I_1, I_2, \phi_1, \phi_2), \tag{5.7}$$

where h is of the form

$$\begin{aligned}h(I_1, I_2, \phi_1, \phi_2) = {}& h_1(I_1, I_2, \phi_1)\\ &- (I_3 - I_2)[h_2(I_1, I_2, \phi_2) + h_{12}(I_1, I_2, \phi_1, \phi_2)]\end{aligned} \tag{5.8}$$

with

$$\begin{aligned}h_1(I_1, I_2, \phi_1) = {}& 4(I_1^2\cos^2 2\phi_1 + I_2^2\sin^2 2\phi_1)\\ &\times \{16(I_3 - I_2)^2[(I_3 - I_2)^2 + I_2^2 - I_1^2] + (I_1^2\sin^2 2\phi_1 + I_2^2\cos^2 2\phi_1)^2\},\end{aligned} \tag{5.9}$$

$$h_2(I_1, I_2, \phi_2) = 4(I_2^2+I_1^2)(I_2^2-I_1^2)^{\frac{1}{2}} \times \{[16(I_3-I_2)^2+I_2^2-I_1^2]\cos 2\phi_2-(I_3-I_2)(I_2^2-I_1^2)^{\frac{1}{2}}\cos 4\phi_2\} \quad (5.10)$$

and

$$h_{12}(I_1, I_2, \phi_1, \phi_2) = g_{12}(I_1, I_2, \phi_1, \phi_2)+g_{12}(-I_1, I_2, \phi_1, -\phi_2) \quad (5.11)$$

where

$$g_{12}(I_1, I_2, \phi_1, \phi_2) = 8(I_3-I_2)(I_2-I_1)^2 I_1 I_2 \cos 4(\phi_1-\phi_2) - 8(I_2-I_1)(I_2^2-I_1^2)^{\frac{1}{2}}[I_1 I_2(I_2-I_1)+4(I_2+I_1)(I_3-I_2)^2] \times \cos(4\phi_1-2\phi_2)-2(I_2^2-I_1^2)^{\frac{3}{2}}(I_2-I_1)^2\cos(8\phi_1-2\phi_2) + 2(I_2^2-I_1^2)(I_3-I_2)(I_2-I_1)^2\cos(8\phi_1-4\phi_2). \quad (5.12)$$

Note that I_3, which is a constant of the motion, appears as a parameter in these expressions. We recall from equation (3.8) that $I_3 = \frac{1}{2}L^2 = J_1+J_2+J_3$ where L^2 is given in terms of the vortex positions by equation (2.8). Note also that the argument of the logarithm, h, acts as an effective Hamiltonian: any equation of motion derived from H involves a logarithmic derivative of h, e.g. the pair

$$\dot{\phi}_1 = \frac{\partial H}{\partial I_1} = -\frac{1}{4\pi h}\frac{\partial h}{\partial I_1}; \quad \dot{I}_1 = \frac{1}{4\pi h}\frac{\partial h}{\partial \phi_1}. \quad (5.13)$$

Now h is constant and so introducing a scaled time variable $t_* = t/4\pi h$ we get

$$\frac{d\phi_1}{dt_*} = -\frac{\partial h}{\partial I_1}, \quad \frac{dI_1}{dt_*} = \frac{\partial h}{\partial \phi_1}. \quad (5.14)$$

Hence for initial conditions with a given value of H we may just as well use h as the Hamiltonian. The form, equations (5.8)–(5.12), derived for h is reminiscent of Hamiltonians that arise in problems with coupled nonlinear oscillators. Such systems often display stochasticity due to the phenomenon of resonance overlap (see Chirikov (1979), and the article by J. Ford in Jorna (1978)). Thus it is eminently plausible, and will be made more so by the numerical work to be described in § 6, that the four-vortex problem considered has chaotic solutions. The individual 'oscillators' in our case are of course given by the pieces h_1 and h_2 of the effective Hamiltonian h while the resonance overlap terms are embodied in the 'interaction' h_{12}. There are several more resonant interaction terms in our Hamiltonian than in the model problems commonly considered in the work referred to above.

The Hamiltonian (5.7)–(5.12) displays a number of discrete symmetries in its independent variables I_1, I_2, ϕ_1, ϕ_2. For example, if I_1 is replaced by $-I_1$ and ϕ_1 by $\frac{1}{2}\pi-\phi_1$ the expression for h is unchanged. Such invariances are related to the invariance of the original problem under arbitrary permutations of vortex indices. Indeed one may calculate the transformation of ζ_1, ζ_2, ζ_3 (and hence of J_n, θ_n or I_n, ϕ_n) induced by any particular permutation. It turns out that the obvious symmetries of H (those that involve sign changes of I_1 or changes of ϕ_1 by $\pm\frac{1}{2}\pi$ or both) are induced by a subgroup D_4 of the full symmetry group S_4. The group D_4 is isomorphic to the symmetry group of a square. Table 5.1 summarizes the results by giving the permutations and the corresponding transformations of I_1 and ϕ_1. (In the notation used in table 5.1 (1432) means a permutation that maps 1–4, 4–3, 3–2 and 2–1.)

TABLE 5.1. INDEX PERMUTATIONS D_4 AND CORRESPONDING TRANSFORMATIONS OF I_1, ϕ_1

(1)(2)(3)(4)	I_1	ϕ_1
(1234)	I_1	$\phi_1 - \frac{1}{2}\pi$
(1432)	I_1	$\phi_1 + \frac{1}{2}\pi$
(12)(34)	$-I_1$	$-\phi_1 + \frac{1}{2}\pi$
(13)(24)	I_1	ϕ_1
(14)(23)	$-I_1$	$-\phi_1 - \frac{1}{2}\pi$
(13)(2)(4)	$-I_1$	$-\phi_1$
(1)(3)(24)	$-I_1$	$-\phi_1$

(b) *A class of exact solutions*

There is a simple discrete symmetry of the problem of four identical vortices which leads to a class of exactly calculable solutions (see the discussion following equation (2.8)). The existence of these solutions was mentioned in Novikov's (1975) paper. It is easily verified from equation (2.1) that if initially $z_3 = -z_1$ and $z_4 = -z_2$ then these relations are preserved by the equations of motion and thus it is only necessary to determine as a function of time the positions z_1 and z_2 of vortices 1 and 2 in order to solve the problem. In terms of the variable ζ_n these special (initial) conditions correspond to $\zeta_2 = 0$ or $J_2 = 0$ or $I_2 = I_3$. In this case

$$\dot{I}_2 = -\frac{\partial H}{\partial \phi_2} = \frac{1}{4\pi h}\frac{\partial h}{\partial \phi_2} = \frac{1}{4\pi h_1}\frac{\partial h_1}{\partial \phi_2} = 0. \tag{5.15}$$

In the last step but one the decomposition (5.8)–(5.12) of h, and the condition $I_2 = I_3$ were used. Thus we have shown that if $I_2 = I_3$ initially, it remains fixed at that value in accordance with the preservation of the discrete symmetry.

Let us now analyse the motion in detail. We introduce canonical variables

$$R_1 = (I_1 + I_3)^{\frac{1}{2}} \cos 2\phi_1, \tag{5.16a}$$

$$P_1 = (I_1 + I_3)^{\frac{1}{2}} \sin 2\phi_1, \tag{5.16b}$$

although for plotting purposes we shall use their scaled counterparts

$$\tilde{R}_1 \equiv R_1/(2I_3)^{\frac{1}{2}}, \quad \tilde{P}_1 \equiv P_1/(2I_3)^{\frac{1}{2}}. \tag{5.17}$$

In terms of these variables the level curves (trajectories) of the effective Hamiltonian for this case (i.e. h of equations (5.8)–(5.12) with $I_2 = I_3$) appear as

$$[4\tilde{R}_1^2(\tilde{P}_1^2 + \tilde{R}_1^2 - 1) + 1][4\tilde{P}_1^2(\tilde{P}_1^2 + \tilde{R}_1^2 - 1) + 1]^2 = \Lambda^2. \tag{5.18}$$

Since $\Lambda \leqslant 1$ the physically meaningful region of the $\tilde{R}_1$, $\tilde{P}_1$-plane is bounded by the unit circle centred at the origin. In figure 2*a* we show several level curves of equation (5.18). We have also labelled the maximum O, the minima T, T′, P and P′ and the

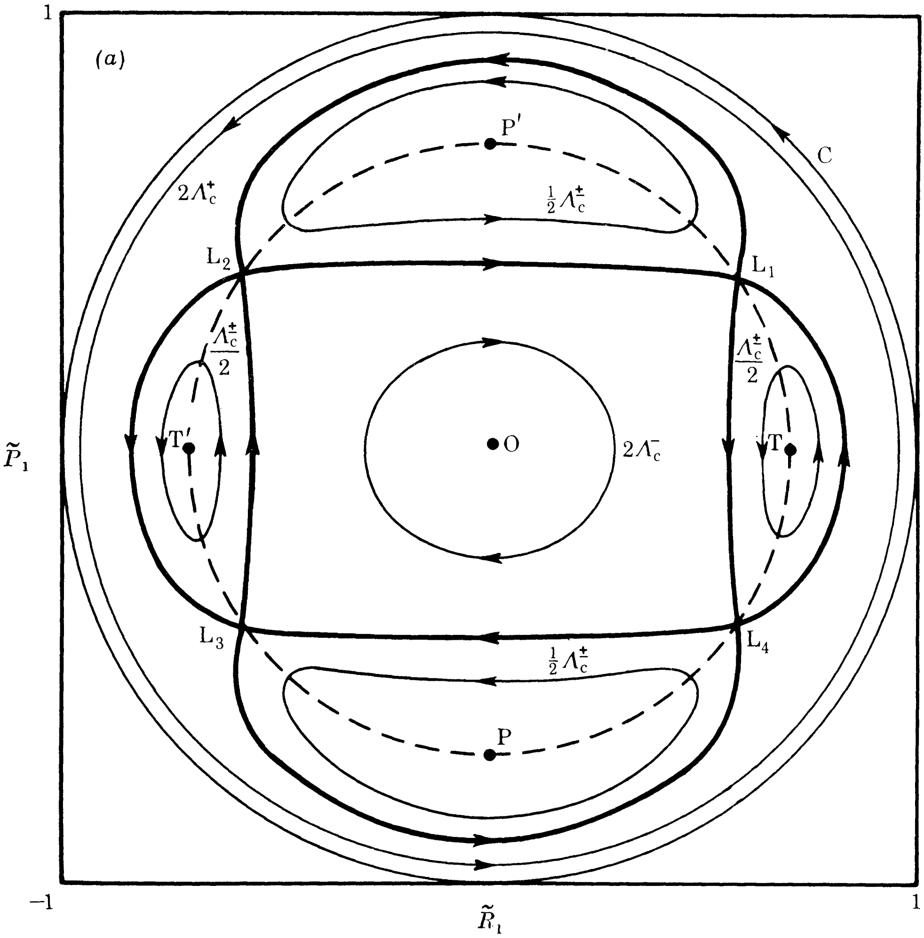

FIGURE 2(*a*). Level curves of the reduced Hamiltonian, equation (5.18), for various values of Λ. The steady-state solutions corresponding to points O, T, T′, P, P′, L_1, L_2, L_3, L_4 and circle C are shown in figure 2*b*. Values of Λ are indicated, superscript plus (minus) means $I_1 > 0$ ($I_1 < 0$). The dashed circle corresponds to collinear configurations.

saddle points L_1, L_2, L_3, L_4 of the function on the left-hand side of (5.18). In figure 2*b* we display scale drawings of the vortex configurations that correspond to these points and to the unit circle C. We note that both O and C correspond to a square configuration with the vortices 1 2 3 4 appearing counter-clockwise for O, clockwise for C. The points T, T′, P, P′ correspond to unrealizable configurations of infinite energy with at least one pair of vortices at the same location. The saddle points $L_1, \ldots, L_4$ correspond to collinear configurations. Collinearity for the special symmetry being discussed here means simply $I_1 = 0$, or in terms of $\tilde{R}_1$, $\tilde{P}_1$,

$$\tilde{R}_1^2+\tilde{P}_1^2 = \tfrac{1}{2}. \tag{5.19}$$

This circle, shown by the dashed line in figure 2*a*, passes through P, P′, T, T′ and $L_1, \ldots, L_4$. Let us mention that for arbitrary N there is a unique (up to permutation of vortices) uniformly rotating configuration with all the (identical) vortices on a line. The positions along the line are given by the zeros of the Nth Hermite polynomial (see Calogero 1977).

A qualitative interpretation of the régimes of motion apparent in figure 2*a* is straightforward and bears a remarkable similarity to the case of three identical vortices (Novikov 1975; Aref 1979). The square configurations (O or C) correspond

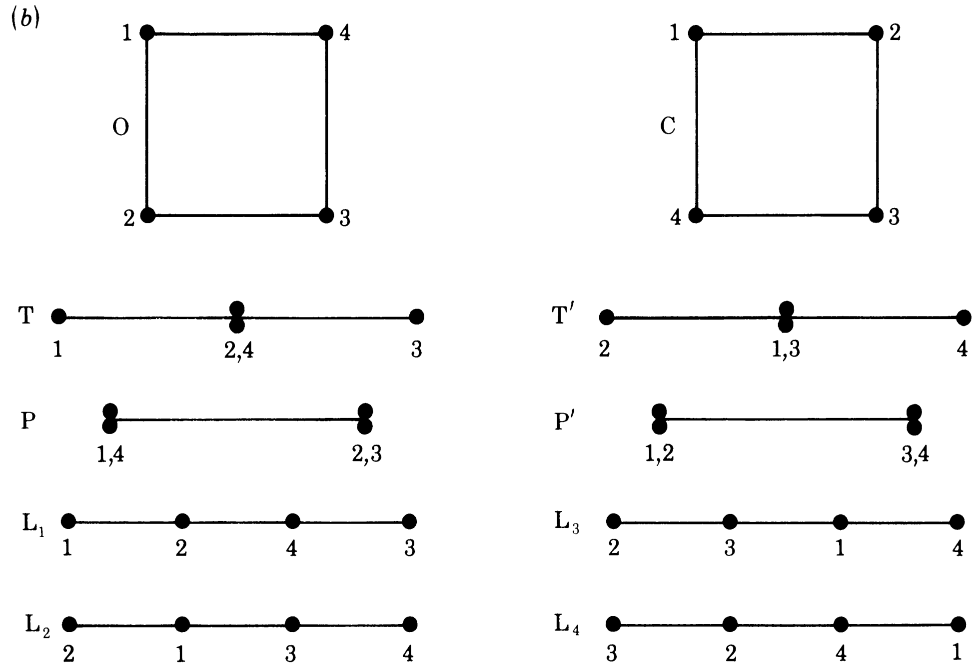

FIGURE 2 (*b*). Real space configurations of vortices corresponding to points O, T, T′, P, P′, L_1, L_2, L_3, L_4 and circle C of figure 2*a*. Units are chosen such that $\kappa = L^2 = 1$.

to $\Lambda = 1$. As Λ is decreased we obtain periodic pulsations of a vortex rectangle. Since the phase trajectory (surrounding O or skirting C) for sufficiently large Λ never intersects the dashed circle, equation (5.19), the vortices never become collinear and the orientation of the configuration is a constant of the motion (corresponding to the non-holonomic constraint that the enclosed area remain positive). For small Λ the motion takes place along a phase trajectory surrounding one of the points P, P′, T, T′, i.e. entirely within one of the four lobes bounded by the separatrix. These trajectories cross the dashed circle twice each period. At each crossing the vortices are collinear but their relative positions on the line are different. On the separatrix that borders these two régimes we find the aperiodic relaxation in infinite time to the steadily rotating collinear configurations (the analogue of the solution (4.17)). In the Appendix we present calculations of the period of pulsation.

The asymptotic collinear vortex states $L_1, \ldots, L_4$ and their saddle connections are of special interest because their behaviour under arbitrary (symmetry breaking) perturbations is intimately tied to the existence of chaotic motion in the general four-vortex problem. M. V. Berry (in an article in Jorna (1978)) and Holmes (1980)

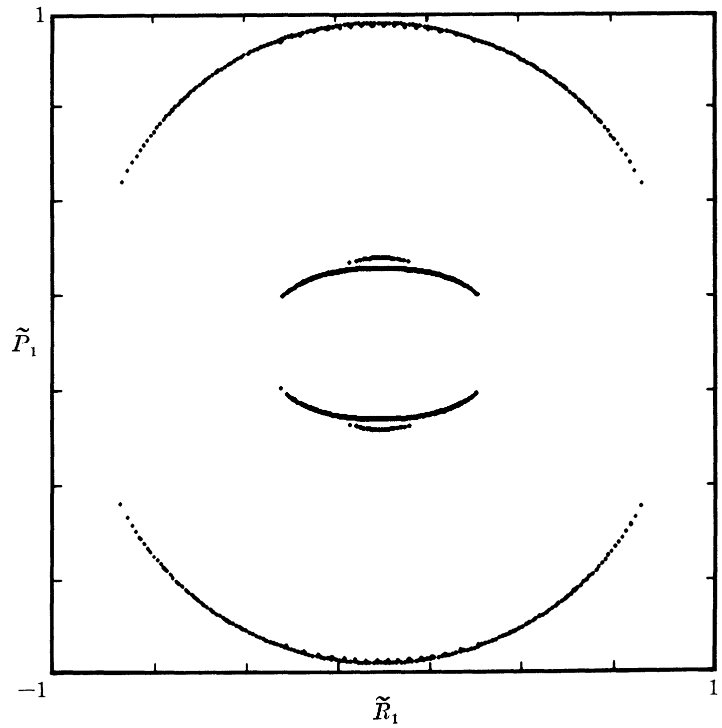

FIGURE 3. Apparently regular Poincaré-section curves for $\Lambda = 0.746245$ ($< 2\Lambda_c$).

review the complicated manner in which the stable and unstable manifolds connecting saddle points can intersect (at an infinite set of heteroclinic points). Phase trajectories must weave through the tangled web of intersections, and sensitive dependence on initial conditions (i.e. chaos) results. The mathematical description of this behaviour is generic for Hamiltonian systems and is by now reasonably well understood. It is however usually difficult to establish analytically the existence of heteroclinic (or homoclinic) points.

6. Numerical experiments

The analysis in § 5 has shown that the effective Hamiltonian describing the relative motion of four vortices has the form that one would expect for a system of two coupled nonlinear oscillators with resonant interactions. Furthermore the exact solutions of § 5 (*b*) revealed the possibility of heteroclinic points in phase space. We should not be surprised, therefore, that Poincaré sections indicating chaotic solutions can be readily produced by numerical simulation. The appearance of such sections with regular 'islands' of tori in a stochastic 'sea' is by now rather well known (see Hénon & Heiles (1964) for an early example or the articles by M. V. Berry, J. Ford and L. J. Laslett in Jorna (1978)).

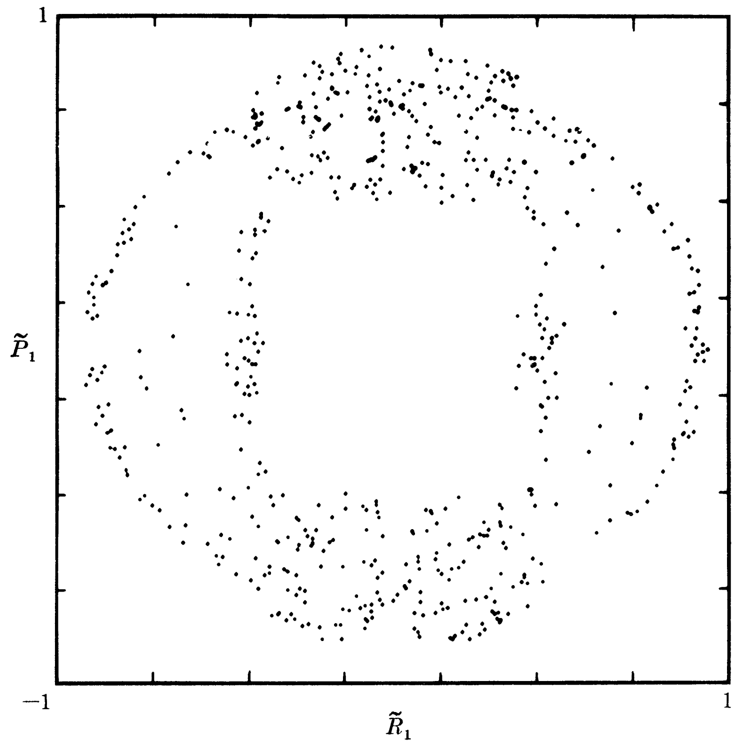

FIGURE 4. Apparently chaotic Poincaré section for $\Lambda = 0.381\,197$ ($< \Lambda_c$).

The Poincaré sections considered here arise directly from the two-degree-of-freedom Hamiltonian of § 5. We define R_1, P_1 and $\tilde{R}_1$, $\tilde{P}_1$ as in equation (5.16) and (5.17) and compute the section $R_2 = 0$, $\dot{R}_2 < 0$ where

$$R_2 = (I_3 - I_2)^{\frac{1}{2}} \sin 2\phi_2 \tag{6.1a}$$

with conjugate variable

$$P_2 = (I_3 - I_2)^{\frac{1}{2}} \cos 2\phi_2. \tag{6.1b}$$

Figures 3 and 4 provide two examples of such sections. The first of these with $\Lambda = 0.746\,245$ shows apparently regular, crescent-shaped tori, the second with $\Lambda = 0.381\,197$ shows chaotic splatter. (The parameter $\Lambda = (\Theta_0^{(4)}/\Theta)^{\frac{1}{2}}$ was defined in § 2.) To put these results in perspective we note that there are just three different types of steadily rotating configurations for four identical vortices. These are the square ($\Lambda = 1$), the face-centred equilateral triangle ($\Lambda = \frac{4}{3}\sqrt{3} \equiv 2\Lambda_c$; see Appendix) and the collinear configurations mentioned in § 5 (*b*) ($\Lambda = \Lambda_c = \frac{2}{3}\sqrt{3}$). Of these the square is stable, the triangle marginally stable and the line unstable. On the basis of these observations Novikov & Sedov (1979*b*; see also Novikov 1980) postulated that the motion of four vortices would be quasiperiodic for

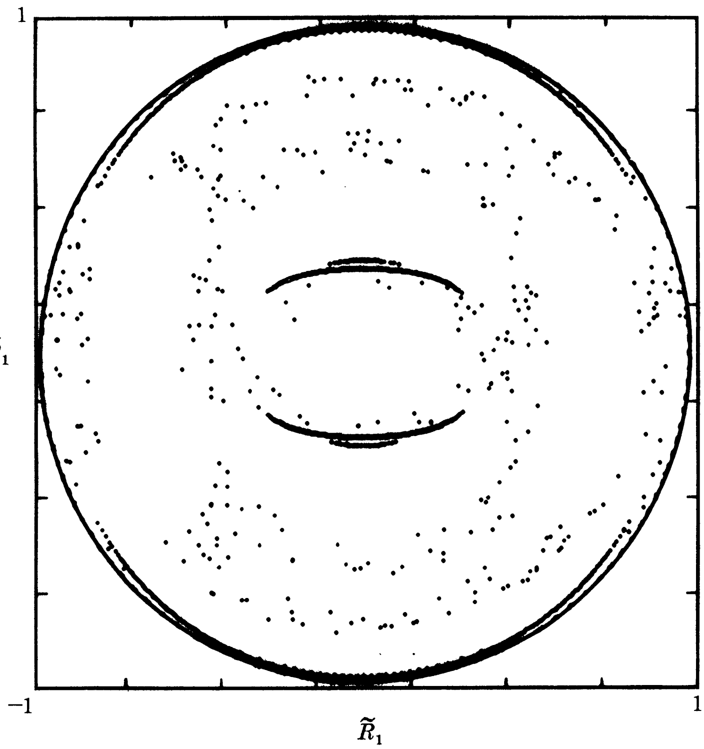

FIGURE 5. 'Islands' of crescent-shaped tori in 'chaotic sea' for $\Lambda = 0.736842$.

$2\Lambda_c < \Lambda < 1$ and chaotic for $\Lambda_c < \Lambda < 2\Lambda_c$. For $\Lambda < \Lambda_c$ they again postulate quasiperiodicity now on the grounds that the available phase space is severely limited. They claim to corroborate aspects of this picture by numerical experiments.

From our point of view the motion of four identical vortices is equivalent to a two-oscillator system which in action-angle variables has a rather conventional but complicated form. Although it is not clear to us whether a sharp onset of chaos (or a sharp disappearance) as the control parameter Λ is decreased is in fact ruled out by general considerations, such behaviour runs counter to the basic framework of understanding that has been constructed around the K.A.M. theorem (cf. J. Ford's article in Jorna (1978)). We believe in accordance with K.A.M. theory that chaotic solutions enter on a set of initial conditions whose measure goes to zero as $\Lambda \rightarrow 1$. Numerical experiments such as the construction of a Poincaré section should then *predominantly* (but not exclusively) detect regular islands of tori for Λ slightly less than 1. The notion that 'large-scale chaos' in the Poincaré section sets in when $\Lambda < 2\Lambda_c$ is probably roughly correct but it is definitely only an approximate criterion. Indeed the smooth section curves of figure 3 correspond to a value of Λ that is somewhat less than $2\Lambda_c(=0.769800)$.

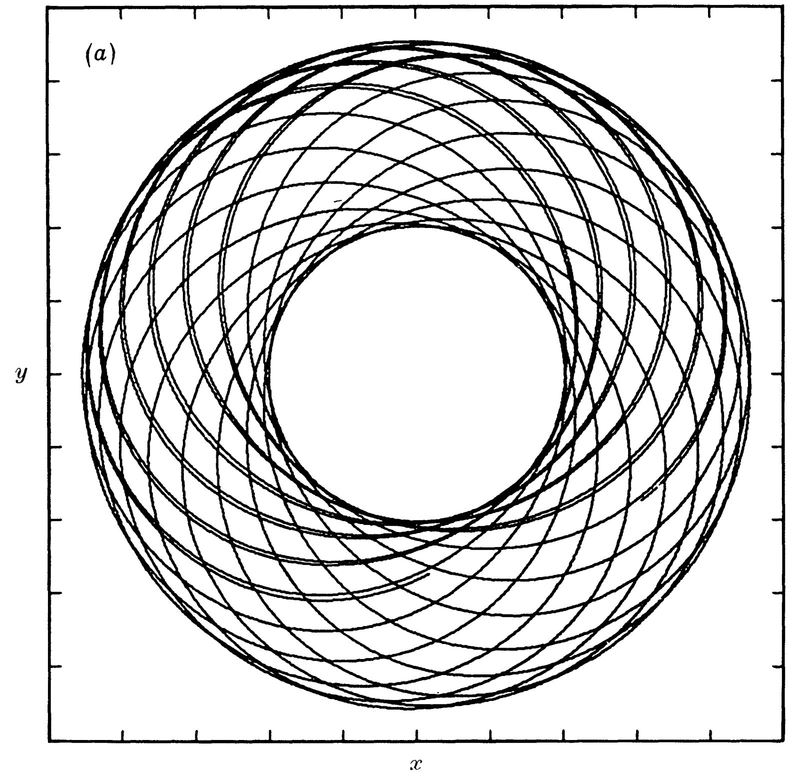

FIGURE 6(a). Trajectory of one vortex for a solution of the type discussed in §5(b). The initial state was a rectangle configuration with aspect ratio 1.75, $\Lambda = 0.597410$.

In similar fashion figure 4 points out the approximate nature of the assumption of quasiperiodicity for $\Lambda < \Lambda_c$: The value of Λ used for that seemingly chaotic section was somewhat less than $\Lambda_c(=0.384900)$. It is worth emphasizing that the argument for more regular motion as Λ decreases is not based on stability considerations but solely on the characteristics of the available phase space region. As the energy is increased for a fixed value of L^2 at least two of the vortices must come together. The four-vortex problem then degenerates to an approximate three- or even two-vortex problem. This follows in the general case much as the trapping in lobes surrounding T, T′ or P, P′ (see figure 2a) for small Λ occurred for the special symmetry in §5(b). Since the two- and three-vortex problems are integrable, approximate integrability of the four-vortex problem should ensue. There is no reason to believe that this should happen suddenly at some specific value of Λ. Moreover by this argument gradual disappearance of stochasticity at high energies would not in general be expected to occur for $N \geqslant 5$ since an effective four-vortex problem could result.

To conclude this discussion we present figure 5 which we believe represents the general situation. All states in this section have $\Lambda = 0.736842$. The crescent shaped tori correspond to slight perturbations of the exact solutions from § 5 (*b*). (The exact solutions of § 5 (*b*) have phase trajectories confined to the section plane as shown in figure 2*a* and cannot be used for the section). The splatter of points around these crescents was generated by a single perturbation of the face-centred triangle configuration adjusted to have the same value of Λ. This coexistence of tori and chaos is in accord with experience from other Hamiltonian systems.

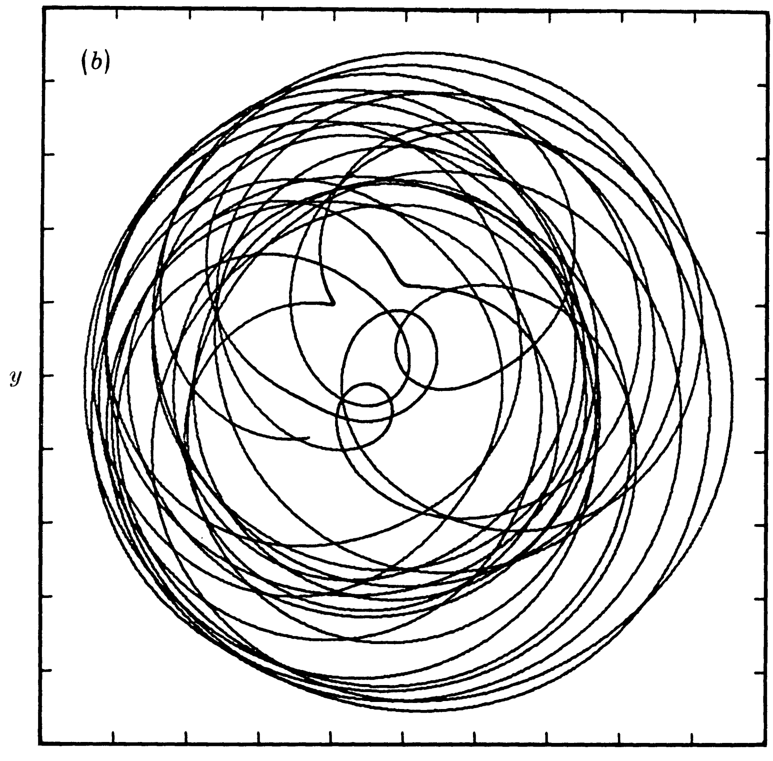

FIGURE 6 (*b*). Trajectory of one vortex for a solution in the chaotic régime. The initial state was a slightly perturbed rectangle configuration with aspect ratio 2, $\Lambda = 0.64$.

Poincaré sections with the use of the vortex separations as independent variables have been published previously (see Aref & Pomphrey 1980*a*; also Novikov & Sedov 1979 and Inogamov & Manakov 1979). Those sections were produced by integrating in time the original vortex equations (2.1), whereas figures 3–5 resulted from integrating the equations of motion derived from the Hamiltonian, equations (5.7)–(5.12). We have continually cross-checked results obtained by the two methods

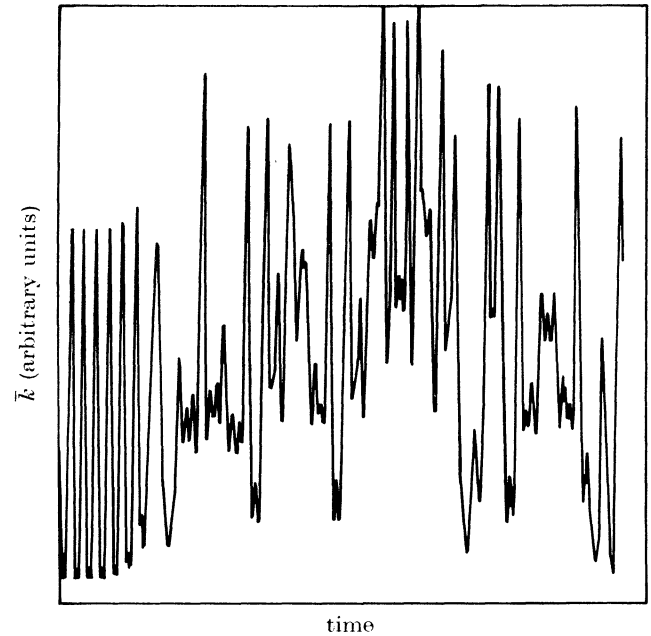

FIGURE 7. Time series of $\bar{k}$, equation (6.2), for a solution in the chaotic régime. The initial condition was as in figure 6*b*.

for consistency. Furthermore runs indicating chaos have been time-reversed to reproduce satisfactorily the initial state from the final one.

Calculation of objects less abstract than a Poincaré section is equally illuminating. Rates of separation for orbits with close initial states have been computed with the expected results (see also Novikov & Sedov 1978). We show in figures 6*a* and *b* the trajectory in real space of one of the vortices. According to § 2 this can also be thought of as the projection of the phase-space trajectory onto the plane spanned by a pair of canonically conjugate variables. The trajectory in figure 6*a* was obtained for one of the solutions in § 5 (*b*). It consists of a regular precession motion bounded by two circles. For $\Lambda > \Lambda_c$ such circles always exist, and the width of the annulus in which the motion takes place is $(L/\sqrt{\kappa})\,(1-\Lambda^2)^{\frac{1}{2}}$. This is easily derived by using the formulae in the Appendix. The trajectory in figure 6*b* on the other hand was obtained for a case in which the Poincaré section appeared chaotic. This trajectory is significantly different from the one in figure 6*a* with loops of many different shapes and sizes. The simple qualitative difference between the paths of a vortex in regular and chaotic motion displayed by figures 6*a* and *b* may be of greater consequence to hydrodynamics than anything else in this paper.

We conclude this section by mentioning one other diagnostic with an interesting physical interpretation. Consider for an assembly of N identical vortices the quantity

$$\bar{k} \equiv \sum_{1 \leqslant \alpha < \beta \leqslant N} |z_\alpha - z_\beta|^{-1}, \tag{6.2}$$

which defines an average wavenumber for the flow. This quantity was introduced by Novikov (1975) who noted that $\bar{k}$ is proportional to the average of all the mutually induced *speeds* within the vortex assembly. Thus, argued Novikov (1975), the phase point describing the state of the vortex assembly will spend a relatively long time in regions of phase space where $\bar{k}$ is small (speeds are low on average). Hence in a time average these low $\bar{k}$-values will receive proportionately greater weight, and statistically an apparent transfer of excitation to low wavenumbers will result. This suggests a mechanistic (albeit qualitative) explanation of a very important characteristic of two-dimensional flow (cf. Fjörtoft 1953; Merilees & Warn 1975).

It was thus considered worthwhile (certain remarks in § 1 notwithstanding) to monitor $\bar{k}(t)$ for initial conditions in the chaotic region. A sample result is shown in figure 7. It is seen that after an initial transient a very complicated time series develops. No particular tendency for $\bar{k}$ to spend a long time in low values is observed. It should be mentioned in this connection that for almost all cases of three-vortex motion and for the solutions of § 5 (*b*) (with $\Lambda \neq \Lambda_c$) $\bar{k}$ will be a periodic function of time, and the arguments given for its decrease fail. Murty & Rao (1970) monitored the variation in time of the mean value of the vortex separations for vortices inside a circular boundary. They report periodic variations of this quantity for $N = 2$ (as expected) but random variations for $N = 3$, 4 and 5. (For $N = 3$ the mean value of the separations and $\bar{k}$ are simply related.) For $N \gtrsim 40$ Murty & Rao (1970) find a smooth monotonic increase in the average separation. Sedov (1976) reports an initial sharp increase followed by a slower systematic decrease in computations of $\bar{k}$ using 100 identical vortices.

7. Discussion

To discuss the implications of our main result, that a system of four identical vortices is non-integrable, we return to the question raised in the Introduction about the integrability of Euler's equation in two dimensions. Renewed interest in this question arose from a brief paper by Hald (1976) in which it was shown that certain truncated Galerkin approximations to the two-dimensional Euler equation with very few degrees of freedom were integrable. As is well known the equations of motion for the Fourier components of vorticity are first-order, ordinary differential equations with quadratic couplings. The models considered by Hald (1976) arise by treating couplings between a chosen set of modes exactly and omitting reference to all other modes. This is a standard procedure for deriving such models. All models have integrals of energy and enstrophy but for integrability also others which Hald (1976) gives explicitly. The question arises whether Galerkin approximations of the size typically used in two-dimensional flow simulations also have as yet undiscovered additional integrals. After all, the two-dimensional Euler equation conserves the vorticity of every fluid particle and thus has an infinity of constants of the motion.

This problem was reexamined by Kells & Orszag (1978) who performed numerical experiments with similar models but with a larger number of modes. The main conclusion of this work was that if truncations in Fourier space are done isotropically, systems with more than about 20 modes behave chaotically. It is presumably still possible to preserve integrability by sparse coupling of wave triads (cf. Meiss 1979). From the point of view of deciding for or against integrability of Euler's equation in two dimensions, truncated Galerkin models are open to the criticism that they destroy most of the circulation integrals of the full continuum equation. The omission of such integrals could in itself destroy integrability.

The main virtue of the point vortex decomposition, which leads to equation (2.1), is precisely that it preserves all circulation integrals by construction while keeping the number of degrees of freedom finite. Furthermore, if the vortices all have strengths of the same sign, the collapse to a point in a finite time mentioned in the Introduction cannot occur. We therefore believe that the present approach can be extended to prove rigorously that Euler's equation with vorticity all of one sign is non-integrable. The vortex decomposition makes use of the important property that if a flow is started at time $t = 0$ with N vortices it will in general continue to contain exactly N vortices. By contrast a flow that initially is represented by a finite number of Fourier modes will in general spread excitation to all modes as time progresses. Since only a finite number of modes are retained in a truncated Galerkin model, errors will inevitably result when the omitted modes should have been excited. In this sense one may argue that the chaos seen in a model such as that of Lorenz (1963) *need not* reflect properties of the Boussinesq equations, whereas the chaotic motion of four vortices does reflect properties of Euler's equation.

Our result that for unbounded flow $N = 4$ is the minimum number of identical vortices needed to produce chaotic solutions can be sharpened in the following way. If we consider instead of the fourth vortex a marker particle, i.e. a 'vortex' of vanishing strength, it has been shown that the motion of this particle as it is advected by the unsteady flow due to the three vortices is non-integrable (Aref & Pomphrey 1980*a*; Ziglin 1980). We shall return to this problem in a subsequent paper. If boundaries or an imposed potential flow are present, the number, N_c, of vortices necessary for chaotic solutions is reduced. For boundaries with no particular symmetry we expect the two-vortex problem to be non-integrable, and a single vortex should suffice to produce chaotic motion of an advected passive marker. Thus N_c is reduced from 4 to 2 (cf. Novikov 1980). For vortices in a half-space bounded by an infinite wall or for vortices in a region bounded by a circle, $N_c = 3$ (Murty & Rao 1970; Novikov & Sedov 1979*b*). Similar arguments applied to the pressure fluctuations at a point on the boundary (which follow from the unsteady Bernoulli equation) suggest that these will be qualitatively different for $N < N_c$ and $N > N_c$. This is of interest in considering the forces on the boundaries in a vortex-dominated flow.

Finally we mention the intriguing fact that there are dynamical systems of a form very similar to the equations we are considering here that are integrable for an arbitrary number of particles. For example the system

$$\dot{z}_\alpha = \frac{\kappa}{2\pi i} \sum_{\beta=1}^{N}{}' (z_\alpha - z_\beta)^{-1}, \tag{7.1}$$

which differs from (2.1) for identical vortices only by the absence of complex conjugation on the left-hand side, is integrable for all N. The reason for this is basically that equations (7.1) are related to the Burgers equation by 'pole decomposition' (cf. Choodnovsky & Choodnovsky 1977) and the Burgers equation can be integrated by using the Cole–Hopf transformation (see Whitham 1974). Alternatively, the solutions to (7.1) are embedded among those of

$$\ddot{z}_\alpha = \frac{1}{2}\left(\frac{\kappa}{\pi}\right)^2 \sum_{\beta}{}' (z_\alpha - z_\beta)^{-3}, \tag{7.2}$$

which is known to be integrable (see the article by J. Moser in Jorna (1978)). It is an interesting question whether one can use the similarity of equations (2.1) and (7.1) and the integrability of the latter to elucidate the properties of a many-vortex system.

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Appendix

We consider quantitative details of the solutions described in § 5 (*b*). Let

$$z_1 = r\,e^{i\vartheta}, \quad z_2 = s\,e^{i\varphi}. \tag{A 1}$$

The positions of vortices 3 and 4 are then given by $z_3 = -z_1$, $z_4 = -z_2$. We assume that all vortices have strength κ. It is clear from equation (2.8) that

$$\kappa(r^2 + s^2) = \tfrac{1}{2}L^2. \tag{A 2}$$

Hence, we may define an angle χ, $0 < \chi < \pi$, such that

$$r = \frac{L}{(2\kappa)^{\frac{1}{2}}} \cos \tfrac{1}{2}\chi, \quad s = \frac{L}{(2\kappa)^{\frac{1}{2}}} \sin \tfrac{1}{2}\chi. \tag{A 3}$$

The constraint of constant energy takes the form

$$\sin \chi (1 - \sin^2 \chi \cos^2 \psi) = \Lambda \tag{A 4}$$

where $\psi = \varphi - \vartheta$, and $\Lambda = (\Theta_0^{(4)}/\Theta)^{\frac{1}{2}}$ was introduced in § 2. The basic differential equation of motion, which is most easily derived by using equation (2.1), is

$$\dot{\chi} = (2\kappa/\pi \Lambda L^2) \sin^2 \chi \sin 2\psi. \tag{A 5}$$

The pair of equations (A 4) and (A 5) allows the problem of relative motion to be reduced to quadratures in this special case. The connection with the canonical formalism employed in § 5 is given by the following formulae (wherein $\kappa = 1$ in accordance with the conventions in § 5):

$$I_1 = -\tfrac{1}{2}L^2 \sin \chi \sin \psi, \tag{A 6}$$

$$I_2 = I_3 = \tfrac{1}{2}L^2 \tag{A 7}$$

and

$$\mathrm{e}^{2\mathrm{i}\phi_1} = (\cos \chi + \mathrm{i} \sin \chi \cos \psi)/(1 - \sin^2 \chi \sin^2 \psi)^{\frac{1}{2}}. \tag{A 8}$$

From (A 5) we may derive a differential equation for $X = \Lambda/\sin \chi$:

$$\frac{\mathrm{d}X}{\mathrm{d}t} = -\frac{\Lambda \cos \chi}{\sin^2 \chi} \dot{\chi} = -\frac{2\kappa}{\pi L^2} \cos \chi \sin 2\psi. \tag{A 9}$$

Squaring this and using (A 4) we obtain

$$(\mathrm{d}X/\mathrm{d}\tau)^2 = (1 - X)(X^2 - \Lambda^2)(\Lambda^2 - X^2 + X^3) \tag{A 10}$$

where

$$\tau = \frac{4}{\pi} \frac{\kappa t}{L^2} \Lambda^{-2}. \tag{A 11}$$

The solution of (A 10) in general leads to a hyperelliptic integral of the first kind. The nature of these solutions may, however, be seen at once if we notice that equations (A 10) resembles the total mechanical energy for an imagined point mass moving in one dimension in a potential

$$\mathscr{V}_\Lambda(X) = (X - 1)(X^2 - \Lambda^2)(\Lambda^2 - X^2 + X^3). \tag{A 12}$$

For the particular motions given by (A 10) the total energy vanishes. The potential $\mathscr{V}_\Lambda(X)$ vanishes at $X = 1$, $X = \pm \Lambda$ and at the real roots of the cubic $\Lambda^2 - X^2 + X^3$. This cubic has three roots X_0, X_1, X_2 which for $0 < \Lambda \leqslant \Lambda_c = \frac{2}{3}\sqrt{3}$ are all real and are given by

$$X_n = \tfrac{1}{3} + \tfrac{2}{3} \cos \tfrac{1}{3}(2n\pi + \delta) \tag{A 13a}$$

with

$$\cos \delta = 1 - 2(\Lambda/\Lambda_c)^2. \tag{A 13b}$$

For $\Lambda < \Lambda_c$ we have $X_1 < X_2 < X_0$. For $\Lambda = \Lambda_c$ the roots X_2 and X_0 coincide. For $\Lambda > \Lambda_c$ the only real root of the cubic, X_1, is negative and thus of no interest since we must have $X \geqslant \Lambda$.

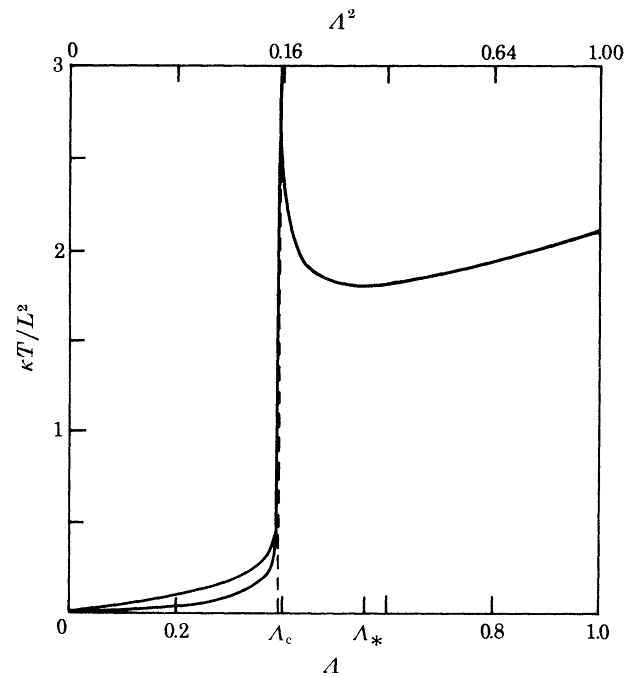

FIGURE 8. The dimensionless period $\kappa T/L^2$, equations (A 15)–(A 16), against Λ (bottom) or Λ^2 (top). This is the period of relative motion for four identical vortices for the special symmetry discussed in §5 (*b*). $\Lambda_c = \frac{2}{3}\sqrt{3}$, $\Lambda_* = 2^{\frac{1}{2}}\Lambda_c$.

A more detailed investigation shows that X oscillates between Λ and 1 for $\Lambda > \Lambda_c$. For $\Lambda < \Lambda_c$ on the other hand X oscillates in one of two intervals: either $\Lambda \leqslant X \leqslant X_2$ or $X_0 \leqslant X \leqslant 1$. The former corresponds to motions in the lobes around points P and P′ in figure 2*a*, the latter to the lobes around T and T′. The separatrix in figure 2*a* arises for $\Lambda = \Lambda_c$, and the corresponding motions are aperiodic. For $\Lambda \neq \Lambda_c$ the period of the relative motion, T, is given by the hyperelliptic integral of the first kind (cf. Byrd & Friedman 1971)

$$\Gamma(\Lambda; a, b) = \int_a^b \mathrm{d}X[(1-X)(X^2-\Lambda^2)(\Lambda^2-X^2+X^3)]^{-\frac{1}{2}} \tag{A 14}$$

as follows: for $\Lambda > \Lambda_c$

$$T = \tfrac{2\sqrt{2}}{3}\pi(\Lambda^2L^2/\kappa)\,\Gamma(\Lambda; \Lambda, 1); \tag{A 15}$$

for $\Lambda < \Lambda_c$

$$T = (\pi\Lambda^2L^2/\kappa)\,\Gamma(\Lambda; \Lambda, X_2), \tag{A 16a}$$

for oscillations about the P, P′-type configurations in figure 2*a* and

$$T = (\pi\Lambda^2L^2/\kappa)\,\Gamma(\Lambda; X_0, 1), \tag{A 16b}$$

for oscillations about the T, T′-type configurations. These formulae produce the correct limiting expressions for T as $\Lambda \to 1$ and $\Lambda \to 0$, respectively. We have evaluated the integral Γ numerically by using Chebyshev polynomials to produce the graphs in figure 8 of the non-dimensionalized period $\kappa T/L^2$ against Λ for the three régimes. The analogous graphs for three vortices appeared as figure 1. We find numerically that the period for $\Lambda > \Lambda_c$ has a minimum at $\Lambda = \Lambda_* = \sqrt{2\Lambda_c} \simeq 0.54433$. W e have no intuitive, mechanistic explanation for the existence of such a minimum. The existence of this minimum in the period seems to preclude any simple correspondence (such as a direct mapping) between the three-vortex problem and this special case of four-vortex motion although, as we have already noted, qualitative features of the relative motion are very similar for the two cases.

The aperiodic case, $\Lambda = \Lambda_c$, which shows up as a singularity of the period in figure 8, can be solved in terms of elliptic integrals of the third kind. The resulting formulae are lengthy and not very informative and will not be reproduced here.

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